Expanders: Overview and Applications

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Overview

What are expanders?

Loosely, expanders are a family of graphs that are:

1. Sparse
2. Highly connected
3. *Explicitly* constructable

3 will be important later...

But for now, we will focus on the expansion properties.

We will consider undirected graphs $G = (V, E)$.

We think of $|V| = n$, where $n \to \infty$ (i.e., graph has many vertices), and G is d-regular, (i.e., $\deg(u) = d$ for all u), where d is a constant.

Note that n grows large, but d remains constant, so G must be sparse.

Something to keep in mind

A random graph is likely to have properties 1 and 2.

What are expanders? "highly connected"

We can define edge expansion using conductance. For some subset of vertices $S \subseteq V$, the conductance $\Phi(S)$ is defined:

$$\Phi(S) = \frac{\text{fraction of edges } (u, v) : u \in S, v \notin S}{\text{fraction of vertices} \in S}$$

The conductance of the whole graph G is similarly defined as:

$$\Phi_G = \min_{0 \leq |S| \leq n/2} \Phi(S) \text{ \textit{i.e., min} } \Phi \text{ \textit{over non-empty subsets} : } |S| \leq n/2$$

G has good expansion if Φ_G is big, e.g. $\Phi_G \geq 0.01$.

Note that $n/2$ is somewhat arbitrary in this example – in some sense harder to have good expansion as $|S|$ gets larger.

So, we may consider expansion up to a fixed constant, i.e. $|S| \leq .01n$.

What are expanders? "highly connected"

There also exists a notion of vertex expansion...

High-level idea is that every "not-too-large" set of vertices has "many" neighbors.

Formally, G is a (K, A)-vertex expander if for all sets S of at most K vertices, the *neighborhood* $N(S)$ is of size at least $A \cdot |S|$, where

$$N(S) := \{u \mid \exists v \in S \text{ s.t.} (u, v) \in E\}$$

Ideally, A should be large; want to get it close to d.

Random constructions

We consider expanders in the context of left d-regular bipartite graphs.

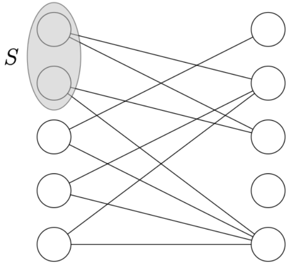
V is partitioned into two sets L and R, d-regularity is maintained on vertices in L, and expansion is chosen from sets $S \subset L$.

[Pinsker '73], by the probabilistic method:

Consider $d \geq 3$, and $n \geq n_0$, where n_0 is some constant. A random left d-regular bipartite graph has, with high probability:

$$|N(S)| \geq (d-2)|S|$$

for all $|S| \leq c_d \cdot n$ and $S \subset L$ (c_d is a constant dependent on d, $\sim \frac{1}{20d^4}$)



Example of a left d-regular bipartite graph

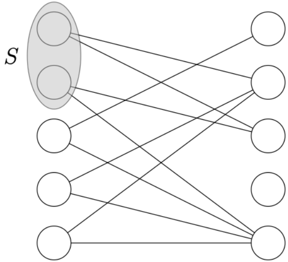
Random constructions cont.

[Bassalygo '81] showed, for a random left d-regular bipartite graph, where $d \geq 64$, $|L| = n$ and $|R| = \frac{3}{4}n$:

$$|N(S)| \geq 0.8 \cdot d \cdot |S|$$

for all $S \subset L$ such that $|S| \leq \frac{.02}{d}n$.

This is again proved by the probabilistic method. The constants 0.8 and 0.02 may vary by proof and application.



Example of a left d-regular bipartite graph

Goal: explicit and strongly explicit constructions

Before we look at one application of expanders, we'll provide definitions and motivation for *explicit* and *strongly explicit* constructions.

An explicit construction of an expander provides, in deterministic poly(n) time, the entire adjacency matrix.

A strongly explicit construction of an expander provides the following in deterministic poly($\log n$) time:

For any $u \in V = [n]$ and $i \in d$, the vertex v which is ith neighbor of u.

Note that a strongly explicit construction does not give the full graph, but rather the specified neighbor of a given vertex in the graph.

Application: deterministic error reduction

Let $\mathcal{A}$ be a randomized algorithm with 1-sided error for a decision problem $\mathcal{P}$, which takes random n-bit string r and x as inputs.
Let's say $\mathcal{A}$ runs in time T.

For instance, if the input x is a number, $\mathcal{P}$ could be "is x prime?", and $\mathcal{A}$ could be the Miller-Rabin primality test.

If $x \in \mathcal{P}$, $\Pr(\mathcal{A} \text{ says } \texttt{YES}) = 1$.

If $x \notin \mathcal{P}$, $\mathcal{A}$ is still correct with high prob, i.e. $\Pr(\mathcal{A} \text{ says } \texttt{YES}) \leq .01$.

Idea for reducing error: Repeat the algorithm d times, with independent random n-bit strings $r_1, \ldots, r_d$.

- Runs in $dT + O(n)$ time, Uses dn random bits, Error prob. $\leq .01^d$

Imagine we want to decrease the error of $\mathcal{A}$, but random bits are *expensive*.

Use a strongly explicit bipartite expander. Say we have an expander according to [Bas81], where $|L| = |R| = 2^{\mathrm{n}}$ (!)

Name vertices in L and R by n-bit strings, rely on strongly explicit property.

$$|N(S)| \geq 0.8 \cdot d \cdot |S|$$

Reducing error: Pick $\ell \in L$ uniformly at random using n random bits. Let $r_1, \ldots, r_d$ be neighbors of ℓ in R, accessible in poly(n) time.

- Runs in poly(n) + dT time
- **Uses n random bits**
- Error probability $\leq \frac{.02}{d}$ (proof coming up!)

Deterministic error reduction conclusion

Claim: Using a bipartite expander, the error probability for $\mathcal{A}$ is $\leq \frac{.02}{d}$.

For any input x, $\mathcal{A}$ is wrong on at most 1% of random seeds. Let $B_x \subset R$ be these "bad strings" in R causing $\mathcal{A}$ to be incorrect.

Let $S \subset L$ be the "bad" choices for $\ell \in L$ – those for which $N(\ell) \subseteq B_x$.

Claim 2: $|S| \leq \frac{.02}{d} 2^n$.
For the sake of contradiction, assume $|S| \geq \frac{.02}{d} 2^n$. Let $S' \subseteq S$ such that $|S'| = \frac{.02}{d} 2^n$. By the expander properties:

$$|N(S')| \geq 0.8d|S'| = .8 \cdot .02 \cdot 2^n = 0.016 \cdot 2^n > 0.01 \cdot 2^n = |B_x|$$

This is a contradiction, because there exists $v \in N(S')$ s.t. $v \notin B_x$, for which $\mathcal{A}$ will answer `NO`.

Thus, the error probability of $\mathcal{A} \leq \frac{.02}{d}$. $\square$

Laplacian of a Graph

- Laplacian: $L = D - A$
- Normalized Laplacian (of a d-regular graph): $L_G = D^{-1/2} L D^{-1/2} = I - \frac{1}{d} A = I - K$
- $\lambda_i(L_G) = 1 - \lambda_i(G)$ ($\lambda_1(G)$: largest eigenvalue of G)
- L_G is PSD. (diagonally dominant and symmetric)
- Smallest Eigenvalue $\lambda_1(L_G) = 0$. Alternatively $\lambda_1(G) = 1$.
- Number of connected components in the graph = dimension of the nullspace of L_G
- $\lambda_2(L_G) > 0 / \lambda_2(G) < 1$ iff graph is connected.

- **Cheeger's inequality:** $\lambda_2(L_G)/2 \leq \Phi_G \leq \sqrt{2\lambda_2(L_G)}$
- Can define **spectral expansion** in terms of $\lambda_2(L_G)$ or $\lambda_2(G)$
- Similar relationship of $\lambda_2(G)$ to vertex expansion exists.

Recall

- Edge Expansion: $\forall |S| \leq \frac{n}{2}, \quad Pr_{u \sim v}[u \in S, v \notin S] \geq \epsilon$
- Conductance: $\Phi_G = \min_{0 \leq |S| \leq n/2} \Phi(S)$
- Vertex Expansion: $\forall |S| \leq \frac{n}{2}, \quad |N(S)| \geq \epsilon \cdot d|S|$

Spectral Expanders

- Definition: An (n, d, ϵ) spectral expander is an n-vertex, d-regular graph with $\lambda_1(L_G) \geq \epsilon$.

Strongly explicit construction

(Gabber and Galil[1981]): Let G be a graph with vertex set $V = \mathbb{Z}_m^2$ and edge set E in which we connect (x, y) to:

- $(x \pm y, y)$,
- $(x \pm (y + 1), y)$,
- $(x, x \pm y)$,
- $(x, y \pm (x + 1))$

Eight neighbors for each vertex, can compute in $O(\log n)$ time.
$(m^2, 8, \epsilon)$ expander with: $\epsilon \approx .1$

Random d-regular graphs

- In a random $n - vertex$, $d - regular$ graph G, with probability $\geq 1 - o_n(1)$, for all $i \geq 2$, we have

$$\lambda_i(G) \leq \frac{2\sqrt{d-1}}{d} + o_n(1)$$

- Would like to get graphs with second eigenvalue close to this
- Certain expander graphs achieve this bound (for certain values of d): Ramanujan Graphs

Ramanujan Graphs

Definition

A connected d regular graph with n vertices that satisfies $\lambda_2(G) \leq \frac{2\sqrt{d-1}}{d}$

Some interesting facts:

- Explicit/strongly explicit constructions exists for certain values of d
- Margulis, Lubotzky--Phillips--Sarnak'88: Infinite sequences of Ramanujan graphs exist for $d =$ prime+1 !
- The constant $2\frac{\sqrt{d-1}}{d}$ in the definition of Ramanujan graphs is asymptotically sharp i.e. $\forall d$ and $\epsilon > 0$, $\exists n$ s.t. all d regular graphs with at least n vertices satisfy $\lambda_2(G) > 2\frac{\sqrt{d-1}}{d} - \epsilon$. [Alon-Bopanna bound]
- Spectral gap is almost as large as possible!
- Ramanujan graphs are essentially the best possible expander graphs!

Quasi-Random Properties of Ramanujan Graphs

Expander Mixing Lemma/ Discrepancy Property

Let G be a d-regular Ramanujan graph. For $X, Y \subseteq V$, let $e(X, Y) = |\{(x, y) \in X \times Y : (x, y), (xy) \in E(G)\}|$. Then,

$$\left| e(X, Y) - \frac{d}{n}|X||Y| \right| \leq 2\sqrt{d|X|Y||}$$

(Edges of the graph are evenly distributed)

- $\lambda_1 = d$ and $\lambda_2 < 2\sqrt{d}$.
- $\bar{e}_1 = \frac{1}{\sqrt{n}}[1, 1, \cdots 1]$
- For $X \subset V$, define $f_X(a) = \begin{cases} 1, & \text{if } a \in X \\ 0, & \text{otherwise} \end{cases}$

- $e(X, Y) = \langle f_X, Gf_Y \rangle$
- Let $f_X = \sum_i a_i \bar{e}_i$ and $f_Y = \sum_j b_j \bar{e}_j$
- Let $a_1 = \langle f_X, \bar{e}_1 \rangle = \frac{|X|}{\sqrt{n}}$ and $b_1 = \frac{|Y|}{\sqrt{n}}$
-

$$\begin{aligned}
\left| e(X, Y) - \frac{d}{n}|X||Y| \right| &= |\langle f_X, Gf_Y \rangle - \lambda_1 a_1 b_1| \\
&\leq |\lambda_2| \sum_{i=2}^{n} |a_i b_i| \\
&\leq |\lambda_2| \|f_X\|_2 \|f_Y\|_2 \leq 2\sqrt{d|X|Y||}
\end{aligned}$$

Application: Largest Eigenvalue of {0,1}-PSD matrix

- Let A be a binary (with $\{0, 1\}$) entries PSD matrix.
- Need to find its largest eigenvalue λ_1
- Let G be the adjacency matrix of an n vertex d regular expander graph where $d = O(\frac{1}{\epsilon^2})$
- Find $R = G \cap A$.
- Let $c(R)$ be the largest connected component of R. Then,

$$\lambda_1 - \epsilon n \leq c(G) \leq \lambda_1$$

References

- Ryan O'Donnell. Lecture on expanders : https://www.youtube.com/watch?v=bONlIjZRJhA
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- Fan Chung and Ronald Graham. Sparse Quasi-Random Graphs: https://mathweb.ucsd.edu/~fan/wp/spar.pdf

Thanks! Questions?