Greedy Spanners in Euclidean Spaces Admit Sublinear Separators

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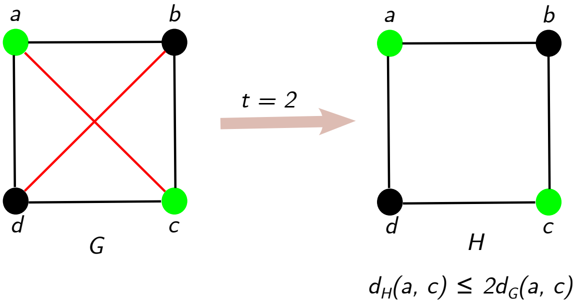
Spanner

Definition

A t-spanner of a graph $G = (V, E, w)$ is a subgraph $H = (V, E_H, w)$ such that for every pair $(u, v) \in V^2$:

$$d_G(u, v) \leq d_H(u, v) \leq t \cdot d_G(u, v)$$

Parameter t is called the stretch of the spanner.



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Given a set of point P in $\mathbb{R}^d$. In our paper, we focus on the spanner of the graph $G = (P, \binom{P}{2}, ||\cdot||_2)$.

Applications

Complexity of Network Synchronization

BARUCH AWERBUCH

Massachusetts Institute of Technology, Cambridge, Massachusetts

(a) Distributed Computing

On Light Spanners, Low-treewidth Embeddings and Efficient Traversing in Minor-free Graphs

Vincent Cohen-Addad[1], Arnold Filtser[2], Philip N. Klein[3], and Hung Le[4]

(b) Approximation Algorithm

EXPLORING PROTEIN FOLDING TRAJECTORIES USING GEOMETRIC SPANNERS

D. RUSSEL and L. GUIBAS

(c) Computational Biology

Near Optimal Multicriteria Spanner Constructions in Wireless Ad-Hoc Networks

Hanan Shpungin, *Member, IEEE,* and Michael Segal, *Senior Member, IEEE*

(d) Wireless Sensor Network

Figure: Applications of spanners

Separator

Definition

A (balanced) separator S is a subset of the vertex set of the graph $G = (V, E)$ such that each connected component of $G[V/S]$ has at most $\frac{2}{3}n$ vertices.

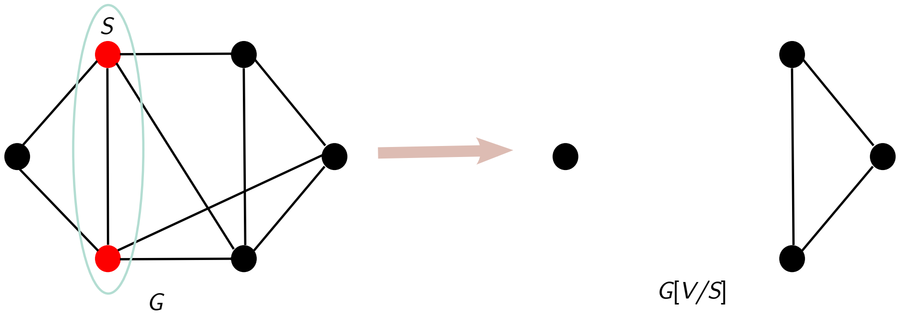

Figure: A separator S of G

Greedy Spanner

Algorithm Greedy($G = (V, E, w), t$)

sort edges in E in increasing order
$H = (V, \emptyset, w)$
for $e = (u, v) \in E$ in sorted order **do**
  if $d_H(u, v) > t \cdot w(u, v)$ **then**
    $E_H = E_H \cup \{e\}$
  end if
end for
return H

Results on Separators of Spanners

- Abam and Har-Peled (2010) constructed a $(1+\epsilon)$-spanner with a separator of size $O(n^{1-1/d})$ for point set in Euclidean space with maximum degree $O(\log^2 n)$.
  - **Open question**: Constructing a spanner with a sublinear separator and a constant maximum degree in metrics of constant doubling dimensions.

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- Recently, Eppstein and Khodabandeh (2021) showed that the greedy spanner for point sets in $\mathbb{R}^2$ admits a separator of size $O(\sqrt{n})$.
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Definition (Informal)

A graph $G = (V, E)$ in a $\mathbb{R}^d$ is τ-lanky if any ball $\mathbf{B}(x, r)$ of radius r is cut by at most τ edges of length at least r.

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Theorem

Let $G = (V, E)$ be an n-vertex graph in $\mathbb{R}^d$ such that G is τ-lanky. Then, G has a balanced separator S such that $|S| = O(\tau n^{1-1/d})$ when $d \geq 2$ and the maximum degree of G is τ.

Contributions

- Introduce τ-lanky - a criterion of graphs admitting sublinear separators and bounded degree.
- τ-lanky implies bounded degree and sublinear separator in strong sense, i.e. every subgraph admits sublinear separator.
- The criterion is simple, can be applied to non-spanner graphs.

Contributions

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  - Greedy spanner has separator of size $O(n^{1-1/d})$. This resolves the open question in Eppstein and Khodabandeh (2021).
- Prove that a spanner in Chan et al. (2016) is $\epsilon^{-O(d)}$-lanky, resolve the open question in Abam and Har-Peled (2010).

Outline

τ – lanky

Spanners in Euclidean Spaces

Spanners in Doubling Metrics

Conclusion

τ-lanky

Definition

A graph $G = (V, E)$ in $\mathbb{R}^d$ is τ-lanky if for any non-negative r, and for any ball $\mathbf{B}(x, r)$ of radius r centered at a vertex $x \in V$, there are at most τ edges of length at least r that are cut by $\mathbf{B}(x, r)$.

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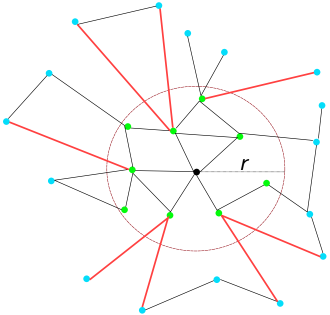

Figure: A ball with radius r is cut by 8 edges with length at least r

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There is nothing special about the length r. In fact, any length cr with c is a positive constant does the trick.

Implications

Lemma

A τ-lanky graph G has maximum degree τ.

Prove by looking at a ball $(u, d_{min}/2)$ for every vertex u of G.

Implications

Theorem

Let G be an n-vertex graph in $\mathbb{R}^d$ such that G is τ-lanky. Then, G has a balanced separator S such that $|S| = O(\tau n^{1-1/d})$ when $d \geq 2$ and $|S| = O(\tau \log n)$ when $d = 1$. Furthermore, S can be found in $O(\tau n)$ expected time.

Proof Sketch

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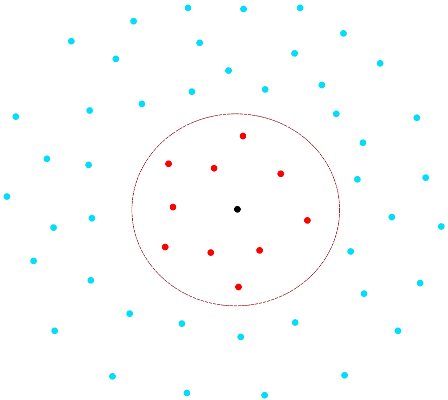
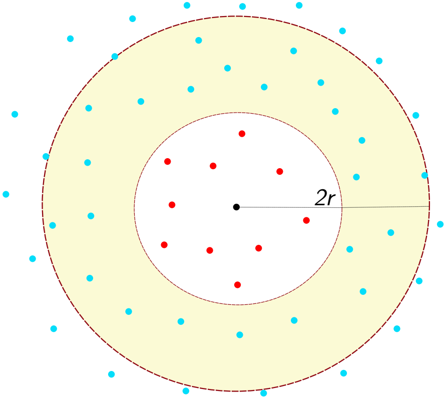
- Find a smallest ball $\mathbf{B}(v, r)$ that contains $\frac{n}{2^{d+1}}$ vertices. Hence, $\mathbf{B}(v, 2r)$ contains at most $\frac{n}{2}$ vertices.



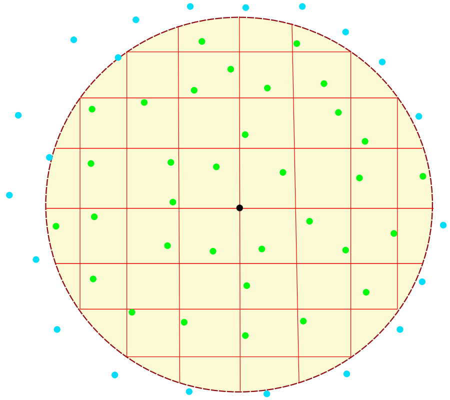
Proof Sketch

- Choose a random radius r^* uniformly in $[r, 2r]$. The expected number of edges with length $rn^{-1/d}$ cutting $\mathbf{B}(v, r^*)$ is $O(n^{1-1/d})$.



Proof Sketch

- For egdes with length in $[rn^{-1/d}, 2r]$, partition $\mathbf{B}(v, 2r)$ into smaller balls accordingly. Each edge cutting $\mathbf{B}(v, r^*)$ must also cut one small ball.

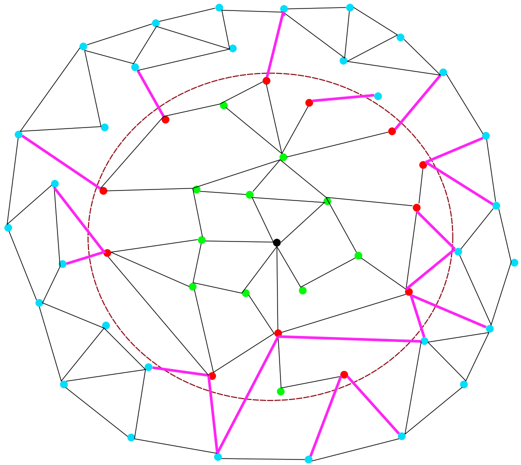


$\#$ edges cut $\mathbf{B}(v, r^*) \leq \tau \times$ ($\#$ balls).

- Number of edges with length larger than $2r$ cutting $\mathbf{B}(v, r^*) \leq \tau$.

Proof Sketch

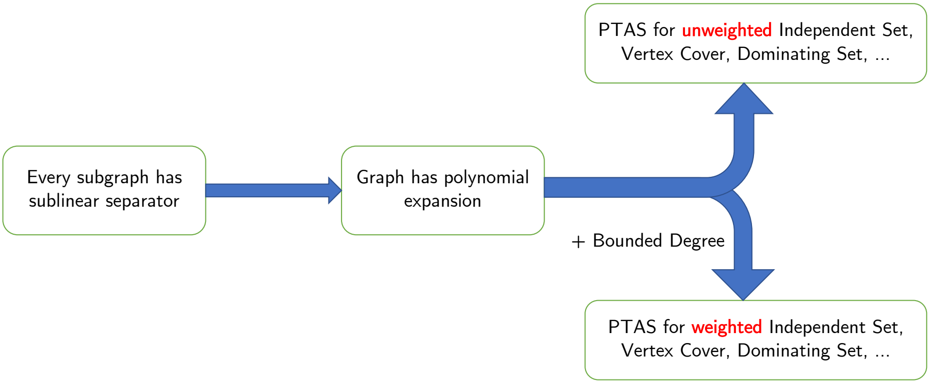
- The separator is the set of vertices inside $\mathbf{B}(v, r^*)$ that incicdent to any edge cutting $\mathbf{B}(v, r^*)$.



Algorithmic Implications

- Unweighted optimization problems such as independent set, vertex cover, dominating set, connected dominating set, packing problems, admit a polynomial-time approximation scheme (PTAS) in a graph has polynomial expansion.
- If the G has bounded degree, then the vertex-weighted version of those problems admit a PTAS.

Algorithmic Implications



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- For any two subsets X and Y of P such that $dist(X, Y) \geq \frac{12}{\epsilon} diam(X)$, there are $O(\epsilon^{1-d})$ edges in G between X and Y.

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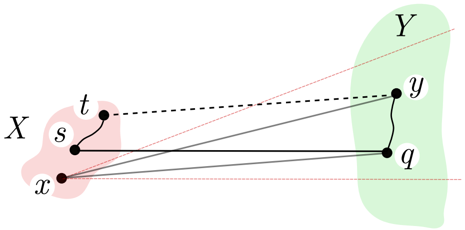

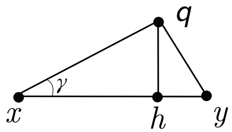

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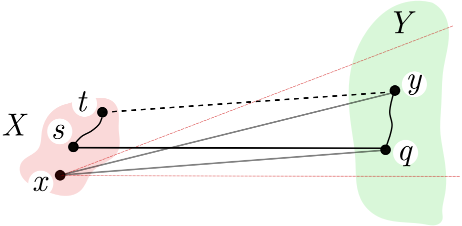

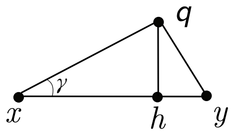

- For any ball **B** of radius r, partition **B** into smaller balls of radius $\epsilon r/48$, then each ball is cut by at most $O(\epsilon^{1-d})$ edges.

Extended Results

Theorem (Bounded fractal dimension)

Let P be a given set of n points in $\mathbb{R}^d$ that has fractal dimension d_f, and G be the greedy $(1+\epsilon)$-spanner of P. Then G has a separator S of size $O(n^{1-1/d_f})$.

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Theorem (Unit ball graphs)

Let G be the greedy $(1+\epsilon)$-spanner of a unit ball graph in $\mathbb{R}^d$. Then G has a separator S of size $O(|V(G)|^{1-1/d})$.

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The Euclidean Space $\mathbb{R}^d$ is (η, d) packable with a constant η. Moreover, the metric space with bounded doubling dimension d is also $(2^{O(d)}, d)$ packable.

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Theorem

Let $G = (V, E, \delta)$ be an n-vertex graph in an (η, d)-packable metric (X, δ) such that G is τ-lanky. Then, G has a balanced separator S such that $|S| = O(\tau n^{1-1/d})$ when $d \geq 2$ and the maximum degree of G is τ.

Spanners in Doubling Metrics

Chan, Gupta, Maggs and Zhou (CGMZ) constructed a $(1+\epsilon)$-spanner with a maximum degree of $\epsilon^{-O(d)}$ for points in doubling metrics of dimension d.

- We prove their spanner is $\epsilon^{-O(d)}$-lanky.

Construction

- Construct a net tree N. In each level i, put all edges with length at most $(4 + \frac{32}{\epsilon})r_i$ to the spanner.

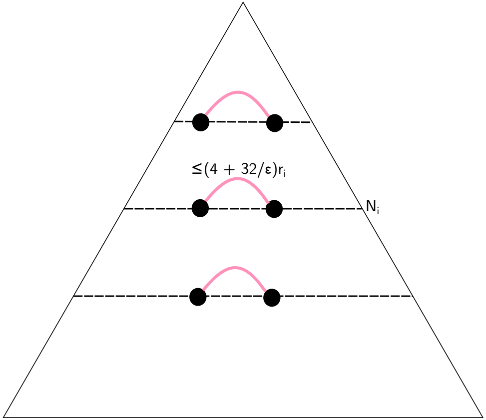

Figure: Net tree spanner

Construction

- Reroute some edges of the spanner such that each edge connects vertices whose levels differ $1/\epsilon$.

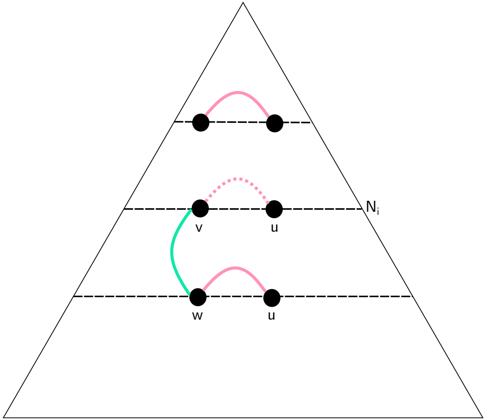

Figure: Net tree spanner after rerouting

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- Given a ball $\mathbf{B}(p, r)$, there are $\epsilon^{-O(d)}$ edges with length $[r, (4+32/\epsilon)r]$ cutting $\mathbf{B}(p, r)$. This is proven by partitioning $\mathbf{B}(p, r)$ into smaller balls.

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- For the edges with length more than $(4 + 32/\epsilon)r$, there are $\epsilon^{-O(d)}$ endpoints of those edges inside $\mathbf{B}(p, r) \implies$ there are $\epsilon^{-O(d)}$ edges cutting $\mathbf{B}(p, r)$ by the bounded degree.

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- The bounded degree spanner in Chan et al. (2016) is $\epsilon^{-O(d)}$-lanky.
- Using the same technique, we proved that the greedy spanner in doubling metric admit separator of size $O(n^{1-1/d} + \log(spread))$ with $spread = \frac{d_{max}}{d_{min}}$.

Open question

Do greedy spanners admit sublinear separators (not depend on *spread*) in doubling metrics?

References

Abam, M. A. and Har-Peled, S. (2010). New constructions of SSPDs and their applications. SoCG'10.

Chan, T. H., Gupta, A., Maggs, B. M., and Zhou, S. (2016). On hierarchical routing in doubling metrics. *ACM Trans. Algorithms*.

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Thank you for listening!

Question?