Sketching Techniques for Hinge Loss

March 28, 2022

Outline

Introduction

Problem Statement

Given n data points $x_1, \ldots, x_n \in \mathbb{R}^d$, a label vector $y \in \mathbb{R}^n$ and f being a classification function

Let $x^* = \arg\min_{x \in \mathbb{R}^d} \sum_{i=1}^{n} f(\langle x_i, x \rangle \cdot y_i)$ and $F(x) = \sum_{i=1}^{n} f(\langle x_i, x \rangle \cdot y_i)$,

Goal: Find a subset of $x'_1, \ldots, x'_r$ points along with corresponding weights $w_1, \ldots, w_r$ s.t. for some small k, we have:

$$F(x') \leq k \cdot F(x^*)$$

where $x' = \arg\min_{x \in \mathbb{R}^d} \sum_{j=1}^{r} w_j \cdot f(\langle x'_j, x \rangle \cdot y'_j)$.

Coresets

Coresets are small subsets of data, often achieved by subsampling from a properly designed distribution.

Deficiencies of coreset constructions:

1. Rely on regularization to obtain small coresets,
2. Usually require random access to data,
3. Require at least two passes over the data (one for calculating/approximating probabilities and the other for subsampling and collecting data),
4. Usually only work in insertion streams, where the data is presented row by row.

Data Oblivious Sketching: What is a linear sketch?

Initialize the data matrix $A \in \mathbb{R}^{n \times d}$ to be a all-zero matrix, where n is large. We have a sequence of updates (i, j, v), each causing a change $A_{ij} = A_{ij} + v$. Updates v of A can be negative.

This is referred to as the **turnstile** model, which is the most flexible dynamic setting.

A **linear sketch** is an algorithm which computes SA as A is updated, where $S \in \mathbb{R}^{m \times n}$ $(m \ll n)$.

Linear sketches support operations such as addition, subtracting and scaling of databases A_j efficiently, since $SA = S \sum_j \alpha_j A_j = \sum_j \alpha_j SA_j$.

Advantages of using Oblivious Sketching:

1. Works well with highly unstructured and arbitrarily distributed data,
2. Allows efficient applications in a single pass of data,
3. Applicable to high velocity streams, since any update can be calculated in $\mathcal{O}(1)$ time,
4. Linear sketches support several useful operations on the data.

Oblivious Sketching for Logistic Loss

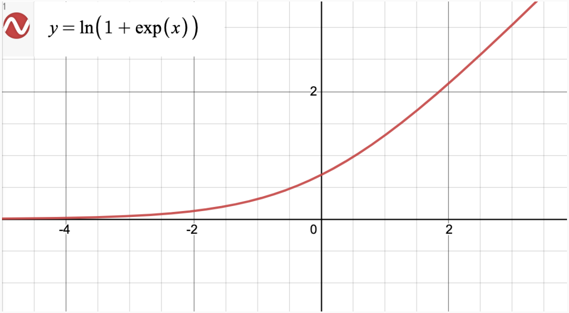
First data oblivious sketch for logistic regression [A. Munteanu, S. Omlor, D. P. Woodruff (2021)]:

- The sketch can be computed in input sparsity time in one pass over a turnstile data stream,
- It reduces the size of a d-dimensional data set from n to *poly*$(\mu d \log n)$ weighted points (where μ is a parameter capturing the complexity of compressing the data),
- It obtains a $\mathcal{O}(\log n)$ approximation to the original problem,
- Can obtain a $\mathcal{O}(1)$ approximation with slight modifications.

Overview of the algorithm

In logistic regression we are given a data matrix $Q \in \mathbb{R}^{n \times d}$ and a label vector $L \in \{-1, 1\}^n$. Let data matrix $A \in \mathbb{R}^{n \times d}$ where each row a_i for $i \in [n]$ is defined as $a_i := -l_i q_i$. Our goal is to find $x \in \mathbb{R}^d$ that minimizes the logistic loss given by

$$f(Ax) = \sum_{i \in [n]} \ln(1 + \exp(a_i x))$$



OVERVIEW OF THE ALGORITHM CONT.

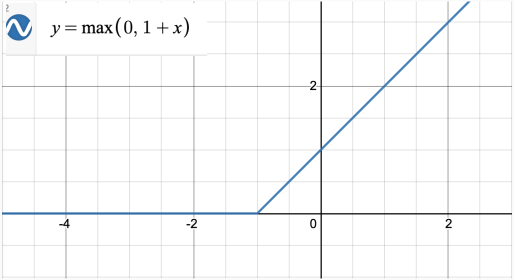
1. Logistic regression loss function can be approximated as $f(Ax) \approx G^+(Ax) + f((Ax)^-)$ which can be handled separately while losing only an approximation factor of 2, where we define:
    - $G^+(y) := \sum_{y_i \geq 0} y_i$ to be the sum of positive entries of y,
    - $(Ax)^-$ the vector Ax with all positive entries replaced with 0
2. The first part $G^+(Ax)$ can be approximated by the collection of sketches $(S_0, \dots, S_{h_{\max}})$
3. The second part $f((Ax)^-)$ can be approximated by a uniform sample (T)

Sketching for Hinge Loss

Overview of the algorithm

In the classification problem we are given a data matrix $Q \in \mathbb{R}^{n \times d}$ and a label vector $L \in \{-1, 1\}^n$. Let data matrix $A \in \mathbb{R}^{n \times d}$ where each row a_i for $i \in [n]$ is defined as $a_i := -l_i q_i$. Our goal is to find $x \in \mathbb{R}^d$ that minimizes the hinge loss given by

$$H = \sum_{i \in [n]} \max(0, 1 + a_i x) = \sum_{a_i x \geq -1} a_i x + 1 = \sum_{y(i) \geq -1} y(i)$$



APPROXIMATING LOSS H

We will approximate H using a matrix S consisting of sketching matrices $S_0, \ldots, S_{l_{\max}}$ where the sketch S_l on each level is presented only a fraction of all coordinates.

This approach is based on a combination of subsampling at different levels and hashing the coordinates assigned to the same level uniformly into a small number of buckets.

Collisions are handled by summing all entries that are mapped to the same level and we use a CountMin-sketch algorithm to recover large enough entries.

ALGORITHM

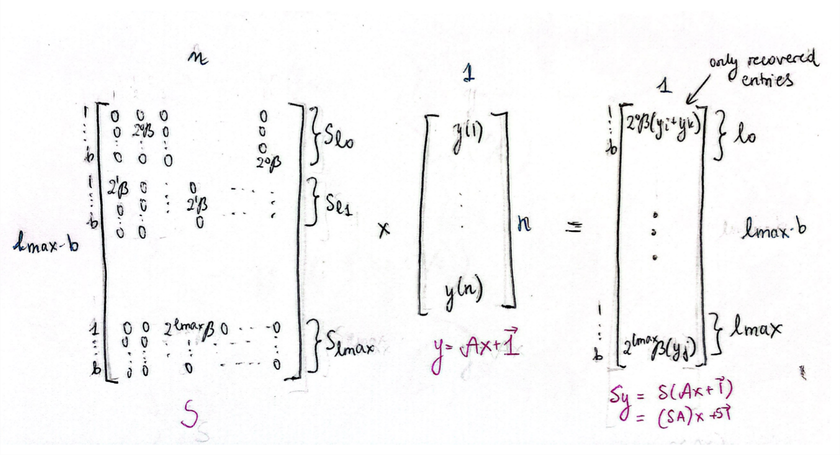
Let b be the number of buckets in each level. We let $l_{\max} = 10\log(\frac{n}{\epsilon})$. So each $S_l \in \mathbb{R}^{(l_{\max}b)\times n}$ for $l \in [l_{\max}]$.

For entry $y(j)$ for any $j \in [n]$

1. $y(j)$ is assigned to level l w.p. $\frac{1}{\beta 2^l}$ where $\beta = 2 - 2^{-l_{\max}}$
2. insert $y(j)$ assigned to level l in a CMS datastructure called C_l with m buckets and using t hash functions ($b = mt$).
3. for each level compute a list of all 'recovered' elements R_l with $\frac{|y(j)-\tilde{y}(j)|}{y(j)} \leq c \cdot \epsilon$ for some small constant c where $\tilde{y}(j)$ are the approximated values by CMS for all $y(j)$ in level l
4. if assigned to level l, $y(j)$ gets the weight:
   - $w_j = \beta 2^l$ if it is recovered ($y(j) \in R_l$)
   - $w_j = 0$ otherwise

Approximate: $H = \sum_{j=1}^{n} y(j)$ by $\tilde{H} = \frac{1}{l_{\max}} \sum_{j=1}^{n} w_j \cdot \tilde{y}(j)$

ALGORITHM CONT.



Preliminaries

Goal of the sketch:

- preserves big entries $y(i)$
- for smaller entries it finds a set of representatives which are in buckets of appropriate weight and are large in contrast to the remaining entries

We will show this by splitting entries $y(i)$ into weight classes and deriving bounds for the contribution of each weight class.

Weight classes: For $q \in \mathbb{N}$ we define $B_q = \{y(j) \mid 2^{-q}H < y(j) \leq 2^{-q+1}H\}$ where $q_{\max} = \log(\frac{n}{\epsilon})$.

Count-Min Sketch guarantees: Let x denote a signal vector. By setting $m = \frac{2}{\epsilon}$, $t = \log(\frac{1}{\delta})$ we have $Pr[\tilde{x}_i - x_i \geq \epsilon \cdot \|x\|_1] \leq (\frac{1}{m\epsilon})^t \leq \delta$ where m denotes the number of buckets and t the number of pairwise independent hash functions.

ANALYSIS

Theorem: $\mathbb{E}[\frac{1}{l_{\max}} \sum_{j=1}^{n} w_j \cdot \tilde{y}(j)] = (1+\epsilon)\mathbb{E}[\sum_{j=1}^{n} y(j)]$

Proof: $\mathbb{E}[\frac{1}{l_{\max}} \sum_{j=1}^{n} w_j \cdot \tilde{y}(j)] =$

$$\frac{1}{l_{\max}} \sum_{j=1}^{n} \sum_{l=0}^{l_{\max}} \frac{1}{\beta 2^l} \cdot \beta 2^l \cdot Pr[y(j) \text{is recovered given it's assigned to level } l] \cdot \tilde{y}(j)$$

Let's compute
$\sum_{j=1}^{n} \sum_{l=0}^{l_{\max}} Pr[y(j) \text{is recovered given it's assigned to level } l] \cdot \tilde{y}(j)$

Let $y(j) \in B_q$ then $y(j) \in (2^{-q}H, 2^{-q+1}H]$. Assume $y(j)$ is assigned to level l. We know that in expectation we have $\frac{n}{\beta 2^l}$ other elements in this level. Lets denote the sum of these elements as
$S_l = \sum_{y(i) \text{ in level } l}^{n/\beta 2^l} y(i) \approx \frac{1}{\beta 2^l} \cdot H$.

Analysis cont.

For $y(j)$ to be recovered correctly we must have that $\frac{|y(j)-\tilde{y}(j)|}{y(j)} \leq c \cdot \epsilon$.
Our CMS datastructure for level l allows us to approximate $\tilde{y}(j) - y(j) \leq \epsilon \cdot S_l$ with high probability. So we have that:

$$\frac{|y(j)-\tilde{y}(j)|}{y(j)} \leq c \cdot \epsilon \implies \frac{\epsilon \cdot S_l}{y(j)} \leq c \cdot \epsilon \quad (1)$$

$$y(j) \geq \frac{S_l}{c} \implies 2^{-q+1} H \geq \frac{1}{c} \cdot (\frac{1}{\beta \cdot 2^l} \cdot H) \quad (2)$$

$$\implies l - q \geq \log(\frac{1}{c\beta}) - 1 \quad (3)$$

$$\implies l \geq q - K \quad (4)$$

for some constant K. Hence $\sum_{j=1}^{n} \sum_{l=0}^{l_{\max}} Pr[y(j)\text{is recovered given it's assigned to level } l] \cdot \tilde{y}(j) \approx \sum_{j=1}^{n} (l_{\max} - q) \cdot \tilde{y}(j)$

ANALYSIS CONT.

Thus we have that

$$\mathbb{E}[\frac{1}{l_{\max}}\sum_{j=1}^{n} w_j \cdot \tilde{y}(j)] = \frac{1}{l_{\max}}\sum_{j=1}^{n}(l_{\max} - q) \cdot \tilde{y}(j) \quad (5)$$

$$= \frac{l_{\max} - q}{l_{\max}}\sum_{j=1}^{n}\tilde{y}(j) \quad (6)$$

$$= \frac{9\log\frac{n}{\epsilon}}{10\log\frac{n}{\epsilon}} \cdot (\sum_{j=1}^{n} y(j) + c\epsilon y(j)) \quad (7)$$

$$= 0.9(1 + c\epsilon) \cdot \sum_{j=1}^{n} y(j) \quad (8)$$

$$= 0.9(1 + c\epsilon) \cdot H \quad (9)$$

References

- Oblivious Sketching for Logistic Regression [A.Munteanu, S.Omlor, D.Woodruff (2021)]
- Coresets for Classification - Simplified and Strengthened [T. Mai, C. Musco, A. B. Rao (2021)]
- On coresets for logistic regression [A. Munteanu, C. Schwiegelshohn, C. Sohler, D. P. Woodruff (2018)]
- Unconditional coresets for regularized loss minimization. [A. Samadian, K. Pruhs, B. Moseley, S. Im, R. R. Curtin (2020)]

Thank you!